STABILITY OF CAPILLARY JETS OF ELASTICOVISCOUS FLUIDS

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The stability of a capillary jet of an elasticoviscous fluid with respect to small axisymmetric perturbations is investigated theoretically for two rheological models. It is shown that preliminary tension makes the jet more stable.

The significant retardation of the dissociation of capillary jets of elasticoviscous fluids as compared with jets of ordinary viscous fluids of comparable viscosity has been established by experiments [1-3]. At the same time a theoretical analysis of the stability of an elasticoviscous fluid jet within the framework of small perturbation theory [1] gives the opposite result, the rate of perturbation growth turns out to be higher for an elasticoviscous fluid than for a Newtonian fluid of the same initial viscosity. It is shown below that this contradiction can be eliminated if the fluid state, storing some preliminary strain, is taken as the unperturbed state. This corresponds to experimental observations according to which the "hyperstability" of elastic fluid jets is manifest after the formation of filaments experiencing strong extraction [1].

§1. Let us examine the problem of the stability of a circular, initially homogeneous jet in a quasi-one-dimensional approximation which corresponds to long-wavelength perturbations.

We have the conservation equations

$$\frac{\partial \rho f}{\partial t}+\frac{\partial \rho f v}{\partial x}=0, \tag{1.1}$$

$$\frac{\partial \rho f v}{\partial t}+\frac{\partial \rho f v^2}{\partial x}=\frac{\partial \sigma f}{\partial x}+\frac{\partial \Pi \alpha}{\partial x} \tag{1.2}$$

(ρ is the fluid density and the x axis is directed along the jet axis).

Let us consider two models of an elasticoviscous fluid, i.e., two kinds of relationships between the stresses and strains.

Let us initially take this relationship in the form used in [4]:

$$\theta\Delta\sigma'/\Delta t-\varepsilon\theta\left[\sigma' \mathbf{e}+\mathbf{e}\sigma'-\frac{2}{3}\,\delta\,\mathrm{Sp}(\sigma'\mathbf{e})\right]+\sigma'=2\eta\mathbf{e}, \tag{1.3}$$

$$\sigma=-p\delta+\sigma'.$$

The single nonzero velocity component in the approximation under consideration is the longitudinal velocity v

$$e=\begin{pmatrix} e & & 0 \\ & -\frac{1}{2}e & \\ 0 & & -\frac{1}{2}e \end{pmatrix},\quad \sigma'=\begin{pmatrix} s & & 0 \\ & -\frac{1}{2}s & \\ 0 & & -\frac{1}{2}s \end{pmatrix}. \tag{1.4}$$

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Using the condition of no external pressure on the side surface of the jet, we have

$$-p-\frac{1}{2}s=-q_\alpha;\quad p=q_\alpha-\frac{1}{2}s, \tag{1.5}$$

$$\sigma=\begin{Vmatrix}\frac{3}{2}s-q_\alpha & & 0\\ & -q_\alpha & \\ 0 & & -q_\alpha\end{Vmatrix}, \tag{1.6}$$

$$q_\alpha=\alpha/a. \tag{1.7}$$

Here q_α is the capillary pressure.

We consequently obtain for s from (1.3)

$$\theta\left(\frac{\partial s}{\partial t}+v\frac{\partial s}{\partial x}\right)-\varepsilon\theta s\frac{\partial v}{\partial x}+s=2\eta\frac{\partial v}{\partial x}, \tag{1.8}$$

$$\sigma=\frac{3}{2}s-q_\alpha. \tag{1.9}$$

Equations (1.1), (1.2), (1.8), and (1.9) form the fundamental system of equations of the quasi-one-dimensional jet motion.

Let us consider small deviations from the fundamental state, corresponding to the state of a relaxing "filament" for which we have

$$v=v_0\equiv 0,\quad s=s_0=S^0\exp(-t/\theta);\quad f=f_0. \tag{1.10}$$

Let the primes denote small deviations of the corresponding quantities from the fundamental state and let us write the equations for the perturbations. We have

$$\frac{\partial f'}{\partial t}+f_0\frac{\partial v'}{\partial x}=0, \tag{1.11}$$

$$\frac{\partial v'}{\partial t}=\frac{1}{\rho}\cdot\frac{\partial\sigma'}{\partial x}+\frac{\alpha}{\rho f_0}\cdot\frac{\partial\Pi'}{\partial x}+\frac{\sigma^0}{\rho f_0}\cdot\frac{\partial f'}{\partial x}, \tag{1.12}$$

$$\sigma'=\frac{3}{2}s'-q'_\alpha, \tag{1.13}$$

$$\theta\frac{\partial s'}{\partial t}+s'-\varepsilon\theta s_0\frac{\partial v'}{\partial x}=2\eta\frac{\partial v'}{\partial x}, \tag{1.14}$$

$$\theta\frac{\partial s'}{\partial t}+s'=2\eta^*\frac{\partial v'}{\partial x};\quad \eta^*=\eta+\frac{1}{2}\varepsilon\theta s_0. \tag{1.15}$$

We shall henceforth be interested in "fast" processes for which the characteristic time $\tau\sim 1/\mu$ is much less than θ. The change in the quantity s_0 in time can hence be neglected, and s_0 and η^* can be considered constants (an analysis of the stability of the "frozen" state).

Let us set

$$\begin{aligned} f'&=f_0Fe^{\mu t}\cos kx, & v'&=Ve^{\mu t}\sin kx,\\ \sigma'&=\Sigma e^{\mu t}\cos kx, & s'&=Se^{\mu t}\cos kx,\\ q'_\alpha&=Qe^{\mu t}\cos kx, & \Pi'&=\Pi e^{\mu t}\cos kx.\end{aligned} \tag{1.16}$$

We hence have from (1.11)-(1.15)

$$\mu F + kV = 0, \quad \Sigma = \frac{3}{2} S - Q,$$

$$\mu V = -\frac{\Sigma}{\rho} k - \frac{\alpha \Pi}{\rho f_0} k - \frac{\sigma_0 k}{\rho} F = -\frac{\Sigma k}{\rho} - \frac{\alpha^*}{\rho a} kF, \tag{1.17}$$

$$(1+\mu\theta) S = 2\eta^* kV; \quad \alpha^* = \alpha + a\sigma_0 = \frac{3}{2} a S_0.$$

We finally have the relationship (a is the jet radius)

$$\Pi = 2\pi a, \quad f = \pi a^2, \quad q_\alpha' = -\alpha \left(\frac{1}{a^2} + \frac{\partial^2}{\partial x^2}\right) a', \tag{1.18}$$

$$Q = -\frac{1}{2} \alpha \left(\frac{1}{a^2} - k^2\right) F.$$

Eliminating F, V, Σ, Π, Q, and S from the relationships (1.17) and (1.18), we obtain the characteristic equation in the form

$$\mu^2 + \frac{3\eta^* k^2 \mu}{(1+\mu\theta)\rho} = \frac{\alpha^* k^2}{\rho a} + \frac{\alpha k^2}{2a\rho} (1 - k^2 a^2), \tag{1.19}$$

which has been obtained for fast processes and can consequently be used only to seek the roots satisfying the inequality $\mu\theta \gg 1$. In this case it can be simplified and results in the form

$$\mu^2 = \frac{\alpha^*\theta - 3\eta^* a}{\rho\theta} k^2 + \frac{\alpha k^2}{2a\rho} (1 - k^2 a^2).$$

According to the formulas presented above

$$\alpha^* - 3\eta^* a/\theta = \frac{3}{2} a s_0 - \frac{3}{2} \varepsilon\theta s_0 - 3\eta a/\theta.$$

We therefore finally obtain

$$\mu^2 = -\frac{3}{2} \cdot \frac{(\varepsilon - 1) s_0 + 2\eta/\theta}{\rho} k^2 + \frac{\alpha k^2}{2a\rho} (1 - k^2 a^2). \tag{1.20}$$

The right side of the relationship (1.20) is negative if

$$3 [(\varepsilon - 1) s_0 + 2\eta/\theta] > \alpha/a. \tag{1.21}$$

Therefore, if the inequality (1.21) is satisfied, fast-growing perturbations (with $\mu\theta \gg 1$) cannot exist and the growth rate of the perturbations is on the order of $1/\theta$ (in other words, viscoelastic effects hence contribute to stabilizing the jet relative to axisymmetric perturbations).

Let us note that according to [4], namely, the case $\varepsilon > 1$ corresponds to the phenomenon of becoming a strand which is typical for elasticoviscous fluids.

§2. Let us consider the same problem within the framework of the rheological model of an elasticoviscous fluid proposed by Leonov [5]. In this case all the distinctions reduce to writing the rheological relationship differently, which has the following form in its simplest version in the Leonov model [5]:

$$\Delta \mathbf{C}/\Delta t - \mathbf{Ce} - \mathbf{eC} = -2\mathbf{Ce}_p, \tag{2.1}$$

$$\sigma + p\delta = 2G\mathbf{C}, \tag{2.2}$$

$$2\theta \mathbf{e}_p = -\left(\mathbf{C} - \frac{1}{3} I_1 \delta\right) - \left(\mathbf{C}^{-1} - \frac{1}{3} I_2 \delta\right). \tag{2.3}$$

Here C is the elastic strain tensor, I_1 and I_2 are its first and second invariants, and δ is the unit tensor.

We have for the quasi-one-dimensional motion under consideration

$$\frac{\partial C}{\partial t}+v\frac{\partial C}{\partial x}-2C\frac{\partial v}{\partial x}\doteq\frac{2}{3\theta}\left(C+C^{1/2}-C^{-1/2}-C^{-1}\right), \tag{2.4}$$

$$\sigma=-q_{\alpha}+2C\left(C-C^{-1/2}\right);\quad C=C_{11}. \tag{2.5}$$

Linearizing these relationships relative to the unperturbed state (C_0, σ_0, q_0, $v = 0$) yields

$$\frac{\partial C'}{\partial t}+\frac{2}{3\theta}\left[2C_0+\frac{3}{2}C_0^{1/2}-\frac{1}{2}C_0^{-1/2}\right]C'=2C_0\frac{\partial v'}{\partial x}, \tag{2.6}$$

$$\sigma'=-q'_{\alpha}+2GC'\left(1+1/2C_0^{-3/2}\right). \tag{2.7}$$

Setting

$$C'=C^{+}e^{\mu t}\cos kx,\quad V'=Ve^{\mu t}\sin kx, \tag{2.8}$$

we have from (2.4) and (2.7)

$$\mu C^{+}+C^{+}/\theta^{*}=2C_0kV;\quad \theta^{*}=3\theta\left[4C_0+3C_0^{1/2}-C_0^{-1/2}\right]^{-1}, \tag{2.9}$$

$$\Sigma=-Q+2GC^{+}\left(1+\frac{1}{2}C_0^{-3/2}\right). \tag{2.10}$$

Comparing (2.9) and (2.10) with the last relationships (1.17), we see that these relationships agree if θ and θ^* are identical and $^2/_3C_0\theta^*G(2 + C_0^{-3/2})$ is taken as η^*. Consequently, we obtain the characteristic equation in the form

$$\mu^2=(\alpha^{*}-3a\eta^{*}/\theta^{*})\,k^2/\rho+\frac{1}{2}\alpha k^2(1-k^2a^2)/a\rho.$$

for the "fast" perturbations. Taking account of the expressions for η^* and θ^* and the relationship (2.5), we have

$$\mu^2=-2GC_0\left(1-2C_0^{-\frac{3}{2}}\right)ak^2/\rho+\frac{1}{2}\alpha k^2(1-k^2a^2)/a\rho. \tag{2.11}$$

Therefore, for sufficiently high initial tensions C_0

$$4C_0G\left(1-2C_0^{-\frac{3}{2}}\right)a>\alpha \tag{2.12}$$

the right side of (2.11) is negative for all wave numbers k and growth of the perturbations turns out to be impossible.

§3. Up to now only one destabilizing factor, the surface tension, was taken into account. The dynamic action of the air can also turn out to be essential for capillary jets moving in air at sufficiently high velocities. Considering the air motion relative to the jet to be potential, then following Weber [6] the appropriate characteristic equation can easily be obtained in the long-wavelength approximation. The component

$$\frac{1}{2}(\rho_1/\rho)\,ak^3f_0(ka)\,U^2, \tag{3.1}$$

is hence added to the right side of the appropriate equation [(1.19) or (2.11)], where ρ_1 is the air density, U is the air velocity relative to the jet, f_0 is the function introduced by Weber (K_0 is the Macdonald function):

$$f_0(\xi)=-K_0(\xi)/K'_0(\xi). \tag{3.2}$$

In the long wavelength domain $ak < 1$, $f_0 < 1$. Hence, the sufficient condition for jet stabilization (neglecting capillary forces) has the following respective forms for the models considered in Secs. 1 and 2:

$$3(\varepsilon-1)\,s_0+6\eta/\theta>\rho_1U^2, \tag{3.3}$$

$$4C_0G\left(1-2C_0^{-3/2}\right)a>\rho_1U^2. \tag{3.4}$$

In other words, elastic tension stabilizes the capillary jet around which air flows, with respect to axisymmetric perturbations if the "elastic" stresses are on the order of the dynamic head of air.

This last result should be considered qualitative since the assumption about the potential nature of the airflow exaggerates the effect of the air (see [7], for instance), however, it yields a correct representation about the necessary order of the stabilizing stresses.

NOTATION

ρ	is the density;
η	is the viscosity;
α	is the surface tension;
θ	is the fluid relaxation time;
a, $f = \pi a^2$, and $\Pi = 2\pi a$	are the radius, area, and perimeter of the jet section;
x	is the longitudinal coordinate;
v	is the longitudinal velocity;
σ	is the axial stress;
σ'	is the stress tensor deviator;
s	is its axial component;
$\Delta/\Delta t$	is the symbol of the Jaumann derivative;
e	is the strain rate tensor;
p	is the pressure;
q_α	is the capillary pressure;
μ	is the perturbation growth increment;
τ	is the characteristic time;
k	is the wave number; the subscript 0 denotes the unperturbed values and the prime denotes perturbations;
G	is the elastic shear modulus;
C	is the elastic strain tensor;
ε	is a dimensionless constant.

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